QUESTION 1. *(2 points)*

Consider the complex numbers $z_1 = 1 + i$, $z_2 = 2 + 3i$, $z_3 = e^{i\,(\pi/2)}$ and $z_4 = e^{i\,(3\pi/2)}$.

(a) Calculate $z_1 \cdot z_2$ and z_1/z_2. *(0.5 point)*

Solution: For $z_1 \cdot z_2$, using $i^2 = -1$.

$$\begin{aligned} z_1 \cdot z_2 &= (1+i)(2+3i) \\ &= 2 + 5i + 3i^2 \\ &= -1 + 5i \end{aligned}$$

For z_1/z_2 we multiply both denominator and nominator with the conjugate of z_2, to get a real number at the denominator:

$$\begin{aligned} z_1/z_2 &= \frac{(1+i)}{(2+3i)} \\ &= \frac{(1+i)}{(2+3i)}\frac{(2-3i)}{(2-3i)} \\ &= \frac{2 - i - 3i^2}{4 - 9i^2} \\ &= \frac{5-i}{13} \end{aligned}$$

Alternative answers: using polar notation for $z_1 = \sqrt{2}e^{i(\pi/4)}$ and $z_2 = |z_2|e^{\theta i}$ where $\theta = \arctan\frac{3}{2}$, and multiplying.

Or using $|ab| = |a||b|$ and $\arg ab = \arg a + \arg b + 2k\pi$ (for all integer k), and $|a/b| = |a|/|b|$ and $\arg a/b = \arg a - \arg b + 2k\pi$.

(b) Express z_3 and z_4 as $\mathrm{Re}(z) + i\,\mathrm{Im}(z)$. *(0.5 point)*

Solution: Use $e^{\theta i} = \cos\theta + i\sin\theta$.

$$\begin{aligned} z_3 &= i \\ z_4 &= -i. \end{aligned}$$

(c) Give the real and imaginary part of $z_5 = (z_1)^6 \cdot (z_4)^4$. *(1 point)*

Solution: Note that $z_1 = \sqrt{2}e^{i\pi/4}$. Hence,

$$\begin{aligned} (z_1)^6(z_4)^4 &= (\sqrt{2}e^{i\pi/4})^6(e^{i3\pi/2})^4 \\ &= (\sqrt{2})^6 e^{i3\pi/2} e^{i6\pi} \\ &= 8 \cdot (-i) \cdot (1) \\ &= -8i, \end{aligned}$$

where in the third step we again used $e^{\theta i} = \cos\theta + i\sin\theta$.

QUESTION 2. *(1.5 points)*

(a) Compute the determinant of the 4×4 matrix A. *(1 point)*

$$A = \begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 3 & 2 & 0 \\ 2 & 0 & 1 & -4 \\ -1 & 2 & 4 & 2 \end{pmatrix}$$

Solution: Row expand along the second row (has the most zeroes and therefore somewhat quicker, but of course any row would work!)

$$|A| = 0 + (-1)^{2+2} m_{2,2} + (-1)^{2+3} m_{2,3},$$

with $m_{2,2}$ and $m_{2,3}$ minors of the $(2,2)$ and $(2,3)$ elements,

$$\begin{aligned} m_{2,2} &= \begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & -4 \\ -1 & 4 & 2 \end{vmatrix} \\ &= 0, \end{aligned}$$

either using Sarrus rule or further row expanding. Moreover,

$$\begin{aligned} m_{2,2} &= \begin{vmatrix} 1 & 2 & -2 \\ 2 & 0 & -4 \\ -1 & 2 & 2 \end{vmatrix} \\ &= 0. \end{aligned}$$

Therefore $|A| = 0 - 0 = 0$.

(b) Does the matrix A have an inverse? Motivate your answer. *(0.5 point)*

Solution: According to the lecture notes, a matrix has a inverse **if and only if** the determinant is non-zero. In this case, A does not have an inverse.

Partial answer: writing $A^{-1} = \frac{1}{\det A} C^T$ with C^T co-factor matrix, and that this might be troublesome when $\det A = 0$.

(Note that this is not actually the full answer! Both C^T and $\det A$ could be zero and hence you can not simply conclude that $\det A = 0$ implies non-existence of the inverse — but luckily this is the result from the lecture notes which you are allowed to state).

QUESTION 3. *(2 points)*

Determine the eigenvalues and eigenvectors of the 3×3 matrix below.

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

Solution: The eigenvalues are the set of real or complex λ such that $B - \lambda I$ is singular, or equivalently, $\det(B - \lambda I) = 0$.

Expanding along the first column (which has the most zeroes),

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)(2-\lambda) - 0 \cdot (-1)) \\ &= (1-\lambda)(2-\lambda)^2 \end{aligned}$$

Therefore $\det(\lambda I - B) = 0$ iff $\lambda = 1, 2$ — the eigenvalues (note that the eigenvalue 2 has multiplicity 2).

For the eigenvectors, we solve the system $Bx = \lambda x$, or equivalently, $(B - \lambda I)x = 0$.

In the case of $\lambda = 1$, (leaving out the row of zeroes for the extended matrix), row-reducing gives

$$
\begin{aligned}
B - I &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Hence for the solution vectors $x = (x_1, x_2, x_3)$, we have that $x_2 = 0$ and $x_3 = 0$, and x_1 is a variable. Hence, an eigenvector for the eigenvalue $\lambda = 1$ is $v = (1, 0, 0)$.

For $\lambda = 2$,

$$
\begin{aligned}
B - 2I &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

For the solution vectors $x = (x_1, x_2, x_3)$ we now have $x_1 = 0$, $x_2 = 0$, and x_3 free — and therefore an eigenvector for the eigenvalue $\lambda = 2$ is $w = (0, 0, 1)$.

QUESTION 4. *(2.25 points)*

Consider the system of linear equations

$$\begin{aligned} x \qquad\quad - 2z &= \lambda + 4 \\ -2x + \lambda y + 7z &= -14 \\ -x + \lambda y + 6z &= \lambda - 12 \end{aligned}$$

where λ is a real parameter.

(a) Solve the system for $\lambda = 0$. How many solutions does the linear system have in this case? *(1 point)*
Solution: Any solution $v = (x, y, z)$ of the system is a solution of $Av = b$, with

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & \lambda & 7 \\ -1 & \lambda & 6 \end{pmatrix}$$

and $b = (\lambda + 4, -14, \lambda - 12)$.
Or, in extended matrix form,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & \lambda + 4 \\ -2 & \lambda & 7 & -14 \\ -1 & \lambda & 6 & \lambda - 12 \end{array}\right)$$

In the case of $\lambda = 0$, after row-reducing,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ -2 & 0 & 7 & -14 \\ -1 & 0 & 6 & -12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that $x = 0$, and $z = -2$, but y is free. Hence, $v = (0, y, -2)$ for any real y — therefore there are infinitely many solutions.

(b) Solve the system for $\lambda \neq 0$. How many solutions does the linear system have in this case? *(1.25 points)*

Solution: For $\lambda \neq 0$, after some row reducing,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & \lambda+4 \\ -2 & \lambda & 7 & -14 \\ -1 & \lambda & 6 & \lambda-12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & \lambda+4 \\ 0 & \lambda & 3 & 2\lambda-6 \\ 0 & \lambda & 4 & 2\lambda-8 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & \lambda+4 \\ 0 & \lambda & 3 & 2\lambda-6 \\ 0 & 0 & 1 & -2 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \lambda \\ 0 & \lambda & 0 & 2\lambda \\ 0 & 0 & 1 & -2 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

In the last step we divided by λ, which is only possible for $\lambda \neq 0$. Thus, for every λ there is a unique solution, $v = (\lambda, 2, -2)$.

QUESTION 5. *(2.25 points)*

Consider the following second-order differential equation:

$$y''(x) + 3y'(x) + 2y(x) = 10\cos(2x)$$

(a) Provide the solution of the associated homogeneous equation. *(1 point)*

Solution: We first calculate the roots of the corresponding quadratic equation $z^2 + 3z + 2 = 0$, which are $z = -2$ and $z = -1$. From the lecture notes therefore the general solution for the homogeneous equation $y''(x) + 3y'(x) + 2y(x) = 0$ are

$$y_h(x) = ae^{-2x} + be^{-x}.$$

for any constants a, b.

(b) Provide the solution of the inhomogeneous equation, with initial conditions *(1.25 points)*:

$$y(0) = 1, \; y'(0) = 0$$

Solution: We first try a particular solution $y_p(x) = c_1 \cdot \cos 2x + c_2 \cdot \sin 2x$, and by plugging this in the ODE we derive $c_1 = -\frac{1}{2}$, $c_2 = \frac{3}{2}$. Therefore any solution is of the form $y(x) = y_h(x) + y_p(x)$, and we calculate the constants $a = \frac{3}{2}, b = 0$ from the initial value conditions. Therefore,

$$y(x) = \frac{3}{2}e^{-2x} - \frac{1}{2}\cos 2x + \frac{3}{2}\sin 2x.$$