TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Examination Differentiaalvergelijkingen en Matrices (2DBA1)
on 29 January 2016, 09.00-12.00 uur.

- Formulate the computations and the results of the exercises in a clear way.
- It is not allowed to use a laptop, graphical calculator or a chart with formulas.
- It is not allowed to use a book or other handwritten material.
- The mobile telephones are not stand by, they are put in a bag. Going to the toilet is only allowed without taking your mobile telephone with you.
- The order in which questions will be resolved is entirely free.
- The exam consists of 6 exercises.

Opmerking: $\exp(x) = e^x$.

1. Given are the complex numbers, with $j^2 = -1$,

$$z_1 = \exp(-\frac{11}{12}\pi j), z_2 = \exp(\frac{2}{3}\pi j).$$

a. Draw the numbers z_1 en z_2 in the complex plane.

b. Let
$$z_3 = (\sqrt{3}+1)((z_1)^2 - (z_2)^2).$$
Calculate the exact values of $\mathrm{Re}(z_3)$ and $\mathrm{Im}(z_3)$.

c. Let
$$z_5 = (\frac{1}{z_1 z_2})^{2016}.$$
Calculate the exact values of $\mathrm{Re}(z_5)$ and $\mathrm{Im}(z_5)$.

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2. Given is the following boundary-value problem:

$$x\,\frac{dy}{dx}(x) + y(x) = x\,\sin(x),\ \text{with}\, x > 0,$$

$$\text{and}\, y(\frac{\pi}{2}) = 0.$$

a) Determine the solution of the homogeneous differential equation.

b) Determine the solution of the given boundary-value problem.

3. Consider the following system of linear equations:

$$\begin{array}{ccccccc} (1+a)\,x_1 & + & x_2 & + & x_3 & = & a+1 \\ x_1 & + & (1+a)\,x_2 & + & x_3 & = & a+3 \\ x_1 & + & x_2 & + & (1+a)\,x_3 & = & -2\,a-4 \end{array}$$

with a parameter a in $\mathbb{R}$.

a) For which value(s) of a has the linear system exactly one solution? (ONLY the values of a are asked and NOT the associated solution.)

b) For which value(s) of a has the linear system no solution?

c) For which value(s) of a has the linear system infinitely many solutions? Calculate the associated solutions, if they exist.

4. The matrix A is defined by:

$$A = \begin{pmatrix} 1 & 5 & 9 & 13 & 6 \\ 2 & 6 & 10 & 14 & 8 \\ 3 & 7 & 11 & 15 & 10 \\ 4 & 8 & 12 & 16 & 12 \end{pmatrix}. \tag{1}$$

a) Determine the rank of A. Is the matrix A invertible?

b) Determine $N(A)$, so determine the null space of A. (Solve $A(\underline{x}) = \underline{0}$.)

c) Determine $R(A)$, so determine the range of A. (Determine the image area of A.)

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5. Given is the matrix:

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & -3 \\ 1 & 0 & 3 \end{bmatrix}$$

a) Calculate the characteristic equation of the given matrix A.

b) Determine the coefficient $\alpha \in \mathbb{R}$, such that:

$$\alpha\, A = -1\,(A^3) + 3\,(A^2).$$

c) Calculate the eigenvalues and the associated eigenvectors of A.

6. Given is the following inhomogeneous linear system of first order differential equations:

$$\frac{d}{d\,t}\underline{y}(t) = A\,\underline{y}(t) + \underline{f}(t), \tag{2}$$

whereby

$$A = \begin{pmatrix} -1 & -3 \\ 1 & 3 \end{pmatrix},\ \underline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ en } \underline{f}(t) = \begin{pmatrix} 22 \\ -4 \end{pmatrix} e^{(3\,t)}.$$

a) Calculate the general solution $\underline{y}_H(t)$ of the homogeneous system of differential equations.

b) Find a particular solution $\underline{y}_P(t)$ of the given inhomogeneous system of differential equations.

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For the exercises, the following number of points can be achieved:

1 a : 2	2 a : 3	3 c : 2	5 a : 2	6 a : 3
1 b : 2	2 b : 4	4 a : 3	5 b : 2	6 b : 3
1 c : 3	3 a : 3	4 b : 2	5 c : 4	
	3 b : 2	4 c : 2		

The result of this exam can be calculated by dividing the total number of points by 4, and it will be rounded to one digit behind the comma. Further the result of the interim test is of importance by calculating the final result of 2DBA0.

If all conditions are fullfilled, then the result will be:

{0.3*(result of the interim test (2DBA2))

0.7*(result of this exam (2DBA1 (> 5.0(!))))}

the result will be rounded to a whole number.

2DBA0 - Answers for Exams

January 24, 2018

Examination DVG of 29 January 2016

1. We have

$$
\begin{aligned}
z_1 &= e^{-\frac{11}{12}\pi i},\\
z_2 &= e^{\frac{2}{3}\pi i},\\
z_3 &= (\sqrt{3}+1)(z_1^2 - z_2^2).\\
z_5 &= (\frac{1}{z_1} z_2)^{2016}.
\end{aligned}
$$

b). Note $z_1^2 = e^{-\frac{22}{12}\pi i} = e^{\frac{1}{6}\pi i} = \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $z_2^2 = e^{\frac{4}{3}\pi i} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Hence, a small calculation gives

$$(\sqrt{3}+1)(z_1^2 - z_2^2) = (2+\sqrt{3}) + i\left(2+\sqrt{3}\right).$$

c). Note that $\frac{1}{z_1 z_2} = e^{\frac{i\pi}{4}}$ and therefore $(\frac{1}{z_1 z_2})^8 = 1$. Since 2016 is divisible by 8, it follows that

$$\left(\frac{1}{z_1 z_2}\right)^{2016} = 1.$$

.

2. We first rewrite the equation to to the standard form,

$$y' + P(x)y = Q(x), \tag{1}$$

where now

$$
\begin{aligned}
P(x) &= \frac{1}{x},\\
Q(x) &= \sin x,
\end{aligned} \tag{2}
$$

and with conditions $x > 0$, $y(\pi/2) = 0$. For the homogeneous case we can set $Q(x) = 0$. Recall from the lectures, we will determine the integrating factor $I = e^{\int P(x)\,dx}$, which in this case is

$$
\begin{aligned}
e^{\int P(x)\,dx} &= e^{\log x}\\
&= x.
\end{aligned} \tag{3}
$$

Multiplying both sides by I and integrating,

$$e^{\int P(x)\,dx}(y' + P(x)y) = e^{\int P(x)\,dx} Q(x), \tag{4}$$

or rewritten as,

$$(e^{\int P(x)\,dx} y)' = e^{\int P(x)\,dx} Q(x), \tag{5}$$

which can be integrated to

$$e^{\int P(x)\,dx} y = \int \left(e^{\int P(x)\,dx} Q(x)\right) dx + C \tag{6}$$

for some constant C, and hence

$$y = Ce^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \int \left(e^{\int P(x)\,dx} Q(x) \right) dx. \tag{7}$$

In our case, the steps will now be

$$(xy)' = x \sin x, \tag{8}$$

hence - using integration by parts to calculate $\int f'g\,dx$ with $f' = \sin x$ and $g = x$.

$$\begin{aligned} xy &= \int x \sin x + C \\ &= -x \cos x + \int \cos x + C \\ &= -x \cos x + \sin x + C, \end{aligned} \tag{9}$$

for some constant C. Hence,

$$y = \frac{C}{x} - \cos x + \frac{1}{x} \sin x. \tag{10}$$

For the homogeneous case we therefore have

$$y = \frac{C}{x}. \tag{11}$$

For the boundary value problem we have $y(\pi/2) = 0$ and use this to determine C:

$$\begin{aligned} y(\pi/2) &= \frac{C}{\pi/2} - \cos \pi/2 + \frac{1}{\pi/2} \sin \pi/2, \\ 0 &= \frac{C}{\pi/2} - 0 + \frac{1}{\pi/2}, \\ C &= -1. \end{aligned} \tag{12}$$

Hence, we conclude that for the bvp the solution is

$$y(x) = -\frac{1}{x} - \cos x + \frac{1}{x} \sin x. \tag{13}$$

3. Let A, y, be defined as

$$A = \begin{pmatrix} a+1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a+1 \end{pmatrix},$$

$$y = \begin{pmatrix} a+1 \\ a+3 \\ -2a-4 \end{pmatrix}.$$

We are considering the system of equations (in matrix form)

$$Ax = y. \tag{14}$$

a). We have (after some calculations) $\det(A) = a^2(a+3)$. Since A is invertible if and only if the determinant is zero, the system has exactly one solution if and only if $a \neq 0, -3$.

Note that for $a = 0$ or $a = -3$ the system either has infinitely many solutions or none at all.

b). For $a = 0$ we have

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}.$$

It is easy to see that for $a = 0$ there is therefore no solution x with $Ax = y$.

c). Finally, consider $a = -3$, in which case the extended matrix is

$$\begin{pmatrix} -2 & 1 & 1 & -2 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 2 \end{pmatrix}.$$

Using Gaussian elimination, one could obtain the matrix

$$\begin{pmatrix} 1 & 0 & -1 & \frac{4}{3} \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in which case the infinite set of solutions is (note that there can be different parametrizations!)

$$\left(\frac{4+3c}{3}, \frac{2+3c}{3}, c \right). \tag{15}$$

4. Matrix rank, null space, and range/image are no longer part of the examination material. For invertibility, note that the matrix is not square. (Note: in the case that A was a square matrix, recall that $\det(A) \neq 0$ if and only if A is invertible.)

5.

a). We have $\chi(\lambda) = \det(A - \lambda I)$, and hence $\chi(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda$.

b). $\alpha = 2$. (Note: for this exercise it was used that $\chi(A)$ is equal to the zero-matrix. This kind of question related to the characteristic equation will no longer be on the exam, or only to be solved via explicit calculation of matrix products)

c). The eigenvalues are the zeroes $\chi(\lambda)$, hence 2, 1 and 0. The eigenvectors are solutions v_i of $Av_i = \lambda_i v_i$ for $i = 1, 2, 3$. By Gaussian elimination and solving the system, this results in, respectively, $(-1, -1, 1)$, $(-2, -1, 1)$, and $(-3, 0, 1)$.

Examination DVG of 6 April 2016

1. We have

$$\begin{aligned} z_1 &= e^{-\frac{1}{3}\pi i}, \\ z_2 &= 2e^{\frac{2}{3}\pi i}, \\ z_3 &= 2z_1^2 - \left(\frac{z_2}{2}\right)^2, \\ z_4 &= (z_1 + z_2)^6, \\ z_5 &= \left(\frac{z_2}{2z_1}\right)^{2017}. \end{aligned}$$

Note that $z_2 = -2z_1$ which will greatly simplify the subsequent calculations.

b). Using $z_2 = -2z_1$, we deduce $z_3 = z_1^2$, and therefore

$$z_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \tag{16}$$

c). Again, using $z_2 = -2z_1$, it follows that $z_4 = (-z_1)^6$ and thus

$$z_4 = 1. \tag{17}$$

with $|z_4| = 1$ and $\arg(z_4) = 0$.

d). Similarly, we have $z_2/(2z_1) = -1$, and hence

$$z_5 = -1. \tag{18}$$

2. (See also the answer for Exercise 2 (1516-Q2) for a small recap on the integrating factor approach). (Moreover, this exercise is *not* representative for the exam. Either the integrating factor will be provided or a relevant helpful substitution - such as trying $y(x) = \frac{f(x)}{\cos x}$ for undetermined f.)

First, rewrite it to the standard form

$$y' + P(x)y = Q(x), \tag{19}$$

with now

$$\begin{aligned} P(x) &= -\tan x, \\ Q(x) &= \frac{2 + e^x}{\cos x}, \end{aligned} \tag{20}$$

where $-\pi/2 \leq x \leq \pi/2$ and $y(0) = \pi$. Hence, the integrating factor is

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{\log \cos x} \\ &= \cos x. \end{aligned} \tag{21}$$

Multiplying both sides by I and integrating,

$$\begin{aligned} y \cos x &= \int \left(\cos x \frac{2 + e^x}{\cos x} \right) dx + C, \\ &= \int (2 + e^x)\, dx + C \\ &= 2x + e^x + C, \end{aligned} \tag{22}$$

for some constant C, and therefore,

$$y = \frac{2x + e^x + C}{\cos x}. \tag{23}$$

The homogeneous solution is

$$y = \frac{C}{\cos x}, \tag{24}$$

and for the boundary-value problem we determine C when taking into account that $y(0) = \pi$.

$$\begin{aligned} \pi &= \frac{2 \cdot 0 + e^0 + C}{\cos 0}, \\ C &= \pi - 1, \end{aligned} \tag{25}$$

and hence

$$y = \frac{2x + e^x + \pi - 1}{\cos x}. \tag{26}$$

3. Let A, y, be defined as

$$A = \begin{pmatrix} 1 & 0 & 1 & b \\ a & 1 & a & (a + ab) \\ b & 0 & (a + b) & (1 + b^2) \\ b & 0 & b & (a - ab + b^2) \end{pmatrix},$$

$$y = \begin{pmatrix} a \\ 1 + a^2 \\ 4 + a \\ 1 + a + ab \end{pmatrix}.$$

We are considering the system of equations (in matrix form)

$$Ax = y. \tag{27}$$

a). Just as before, by calculating the determinant we find $\det(A) = a^2(b-1)$. Since A is invertible if and only if the determinant is zero, the system has exactly one solution if and only if $a \neq 0$ and $b \neq 1$.

b). For $a = 0$, we have as the extended matrix

$$\begin{pmatrix} 1 & 0 & 1 & b & 0 \\ 0 & 1 & 0 & 0 & 1 \\ b & 0 & b & b^2+1 & 4 \\ b & 0 & b & b^2 & 1 \end{pmatrix},$$

Using either Gaussian elimination - or multiplying the first row by b and subtracting from the fourth - we find that for this choice the system has no solution.

Similarly, we can derive that for $b = 1$ there exists no solution.

c). This does not happen.

d). We have in this case

$$A^{-1} = \frac{1}{a^2} \begin{pmatrix} 2 - 2a + a^2 & 0 & -a & 2a - 1 \\ -a^2(a+2) & a^2 & 0 & a^2 \\ -2(a+1) & 0 & a & 1 \\ 2a & 0 & 0 & -a \end{pmatrix}.$$

e). This kind of question will not be on the exam.

4.

a). The eigenvalues are -4, 3 and 0, with respective eigenvectors $(-1,2,1)$, $(-2,-3,2)$, and $(-1,-6,13)$.

b). $\alpha = -1$, $\beta = 12$ (Note, this is no longer part of the material, see Ex. 5 of the exam above)

c). Recall that the columns of a similarity matrix S consists of the eigenvectors, such that the diagonal elements of the the diagonal matrix D consists of the respective eigenvalues. For example,

$$S = \begin{pmatrix} -1 & -2 & -1 \\ 2 & -3 & -6 \\ 1 & 2 & 13 \end{pmatrix},$$

$$D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that we could have chosen any ordering of the eigenvectors in constructing S, as long as it the same order of eigenvalues used in D.